Calc 2 - unit 8A - Day 6

Trig Substitutions to eliminate $\sqrt{\ }$'s

Consider:

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta.$$

$x = \sin\theta$

now look at:

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta \, d\theta}{\cos\theta} = \int d\theta = \theta + C = \arcsin x + C \quad \checkmark$$

$x = \sin\theta$

$dx = \cos\theta \, d\theta$

$\theta = \arcsin x$

Consider:

$$\sqrt{a^2-u^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

$u = a\sin\theta$

now look at:

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta \, d\theta}{2\cos\theta} = \int d\theta = \theta + C = \arcsin\left(\frac{x}{2}\right) + C \quad \checkmark$$

$a = 2 \quad u = 2\sin\theta = x$

$du = 2\cos\theta \, d\theta = dx$

$$\int \frac{dx}{\underbrace{x^2\sqrt{9-x^2}}} = \int \frac{3\cos\theta \, d\theta}{9\sin^2\theta \cdot 3\cos\theta} = \frac{1}{9}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{9}\int \csc^2\theta \, d\theta$$

$a = 3 \quad u = x = 3\sin\theta$

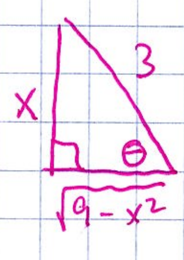
$dx = 3\cos\theta \, d\theta$

$$= -\frac{1}{9}\cot\theta + C$$

$$= -\frac{1}{9} \cdot \frac{\sqrt{9-x^2}}{x} + C \quad \checkmark$$

$\sin\theta = \frac{x}{3}$

$\cot\theta = \frac{\sqrt{9-x^2}}{x}$



Consider:

$$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

$x = \tan\theta$

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$$

$x = \sec\theta$

<u>more generally:</u>

$$\boxed{\begin{array}{l}\text{If } u = a\tan\theta \\ \text{then } \sqrt{a^2+u^2} = a\sec\theta\end{array}} \qquad \boxed{\begin{array}{l}\text{If } u = a\sec\theta \\ \text{then } \sqrt{u^2-a^2} = a\tan\theta\end{array}}$$

ex) $\displaystyle\int \frac{dx}{\sqrt{9+4x^2}} = \int \frac{3/4\sec^2\theta\, d\theta}{3\sec\theta} = \frac{1}{4}\int \sec\theta\, d\theta$

$a = 3 \quad u = 4x = 3\tan\theta$

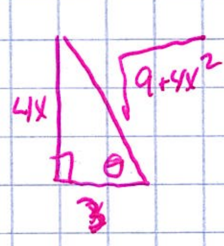
$x = 3/4\tan\theta$

$dx = 3/4\sec^2\theta\, d\theta$

$\displaystyle = \frac{1}{4}\ln|\sec\theta + \tan\theta| + C$

$\tan\theta = \frac{4x}{3}$

$\sec\theta = \frac{\sqrt{9+4x^2}}{3}$



$\displaystyle = \frac{1}{4}\ln\left|\frac{\sqrt{9+4x^2}}{3} + \frac{4x}{3}\right| + C$ ✓

$\displaystyle = \frac{1}{4}\ln\left|\sqrt{9+4x^2} + 4x\right| + C$

ex) $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx = \int \frac{\overbrace{\sqrt{3}\sec\theta}^{x} \cdot \overbrace{\sqrt{3}\sec\theta\tan\theta\, d\theta}^{dx}}{\sqrt{3}\tan\theta} = \sqrt{3}\int \sec^2\theta\, d\theta$

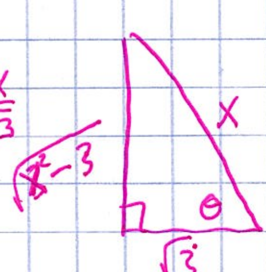
$a = \sqrt{3} \quad u = x = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec\theta\tan\theta\, d\theta$

$= \sqrt{3}\tan\theta + C$

$\sec\theta = \frac{x}{\sqrt{3}}$

$\tan\theta = \frac{\sqrt{x^2-3}}{\sqrt{3}}$



$\displaystyle = \sqrt{3} \cdot \frac{\sqrt{x^2+3}}{\sqrt{3}} + C$

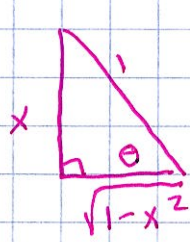
$= \sqrt{x^2+3} + C$ ✓

ex) $\displaystyle\int \frac{dx}{1-x^2} = \int \frac{\cos\theta\, d\theta}{\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \int \sec\theta\, d\theta$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$\sin\theta = \frac{x}{1}$



$= \ln\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C$

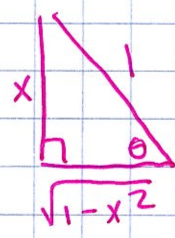
$$\boxed{= \ln\left|\frac{1+x}{\sqrt{1-x^2}}\right| + C}$$

ex) $\displaystyle\int \sqrt{1-x^2}\, dx = \int \cos\theta \cdot \cos\theta\, d\theta = \int \cos^2\theta\, d\theta$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$

$= \displaystyle\int \frac{1+\cos 2\theta}{2}\, d\theta$

$\sin\theta = \frac{x}{1}$



$= \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta$

$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$ $\quad \sin 2\theta = 2\sin\theta\cos\theta$

$= \frac{1}{2}\theta + \frac{1}{4} 2\sin\theta\cos\theta + C$

$$\boxed{= \frac{1}{2}\arcsin x + \frac{2}{4}(x)\sqrt{1-x^2} + C}$$